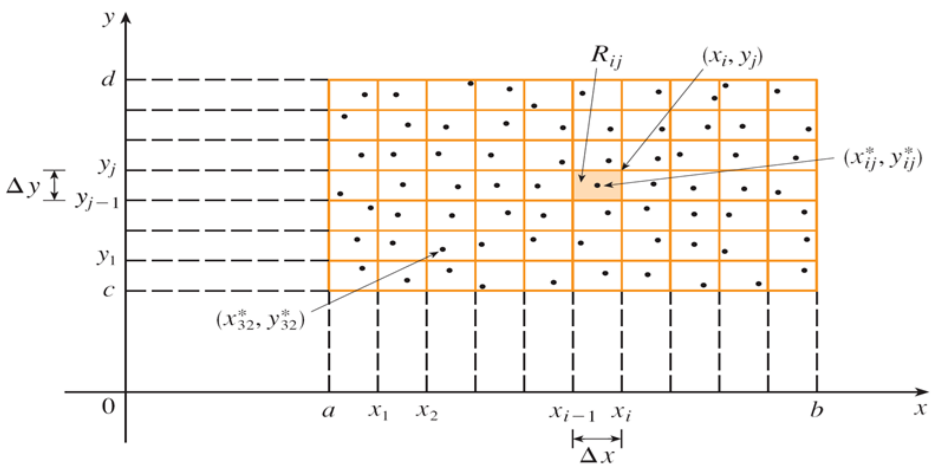
Multiple Integrals

Definition Let $R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \le x \le b,\ c \le y \le d\}$ be a closed rectangle and let f be a function defined on R. Then the double integral of f over R, denoted $\iint_R f\, dA$ or simply $\int_R f\, dA$, is defined by

$$\begin{aligned}\iint_R f(x,y)\,dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A \quad \text{if the limit exists,}\\ &\Longleftrightarrow \text{For any } \varepsilon > 0 \text{ there is an integer } N \text{ such that}\\ &\quad\ \text{if } m,\, n \ge N, \text{ then } \left|\iint_R f(x,y)\,dA - \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A\right| < \varepsilon\end{aligned}$$

where

- the interval $[a, b]$ is divided into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/m$ and the interval $[c, d]$ is divided into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d-c)/n$,
- the rectangle R is divided into $m \times n$ subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ of equal area $\Delta A = \Delta x \Delta y$,
- (x_{ij}^*, y_{ij}^*) is an arbitrary point in R_{ij} and $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$ is called a double Riemann sum of f on R.



Definition A function f is called integrable on R if f is bounded on R and the limit of double Riemann sum exists. [Recall that f is bounded on R if there is a constant M such that $|f(x, y)| \le M$ for all $(x, y) \in R$]

Theorem Let f be a bounded function defined on $R = [a, b] \times [c, d]$. If f is continuous on $R = [a, b] \times [c, d]$ except on possibly a finite number of smooth curves, then f is integrable on R.

Fubini's Theorem If f is continuous on $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy,$$

where $\int_a^b \int_c^d f(x,y)\,dy\,dx$ and $\int_c^d \int_a^b f(x,y)\,dx\,dy$ are called iterated integrals defined by

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx$$

and $\int_c^d f(x,y)\,dy$ means that $f(x,y)$ is integrated with respect to y from $y=c$ to $y=d$ while x is held fixed (as a constant).

Examples Compute each of the following double integrals over the indicated rectangles.

(1.) $\iint_R 6xy^2\,dA,\ R=[2,4]\times[1,2]$.

(2.) $\iint_R 2x-4y^3\,dA,\ R=[-5,4]\times[0,3]$.

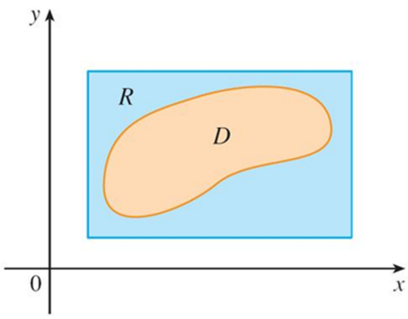
(3.) $\iint_R x^2y^2+\cos(\pi x)+\sin(\pi y)\,dA,\ R=[-2,-1]\times[0,1]$.

(4.) $\iint_R \frac{1}{(2x+3y)^2}\,dA,\ R=[0,1]\times[1,2]$.

(5.) $\iint_R xe^{xy}\,dA,\ R=[-1,2]\times[0,1]$.

Definition Let $D\subset\mathbb{R}^2$ be a bounded subset, $f:D\to\mathbb{R}$ be a function defined on D, $R=[a,b]\times[c,d]=\{(x,y)\in\mathbb{R}^2 \mid a\le x\le b,\ c\le y\le d\}$ be a closed rectangle containing D, and let F be a function on R defined by

$$F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y)\in D \\ 0 & \text{if } (x,y)\in R\setminus D \text{ i.e. } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

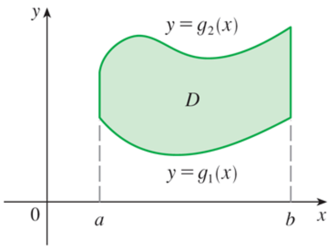


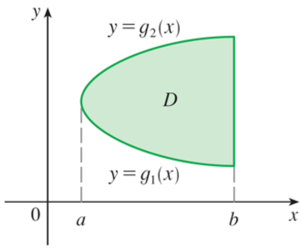
Then we say that f is integrable on D if F is integrable on $R=[a,b]\times[c,d]$. If F is integrable over R, then we define the double integral of f over D by

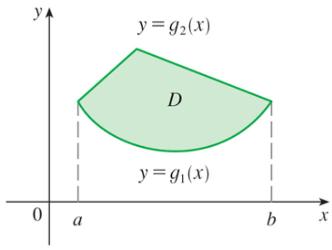
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.$$

Example Let D be a plane region bounded between the graphs of two continuous functions of x, that is,

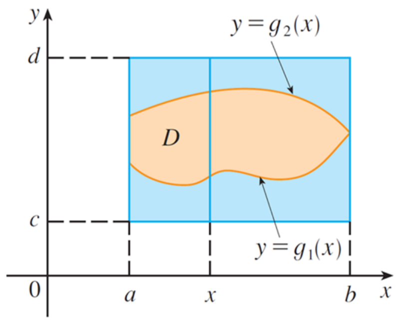
$$D=\{(x,y)\mid a\le x\le b,\ g_1(x)\le y\le g_2(x)\} \quad \text{where } h_1,\ h_2 \text{ are continuous on } [a,b].$$







In order to evaluate $\iint_D f(x,y)\,dA$, we choose a rectangle $R = [a,b] \times [c,d]$, and use the Fubini's Theorem to obtain that



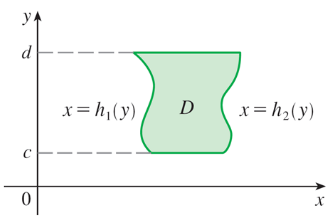
$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA = \int_a^b \int_c^d F(x,y)\,dy\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

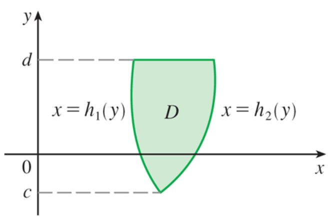
Theorem Let $D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, where g_1 and g_2 are continuous functions on $[a,b]$. If f is a continuous function on D, then f is integrable on D with the integral of f on D given by

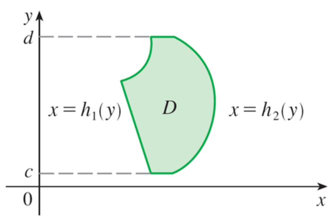
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

Example Let D be a plane region bounded between the graphs of two continuous functions of y, that is,

$$D = \{(x,y) \mid c \le x \le d,\ h_1(y) \le x \le h_2(y)\}, \quad \text{where } h_1,\ h_2 \text{ are continuous on } [c,d].$$



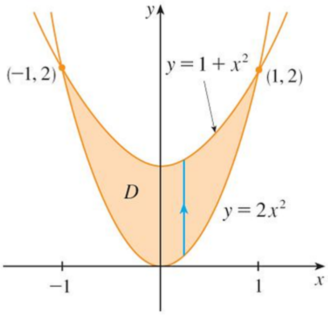


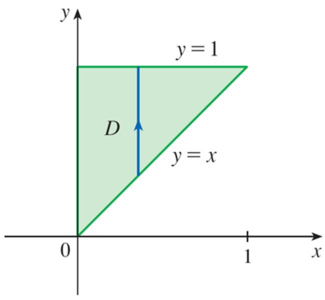


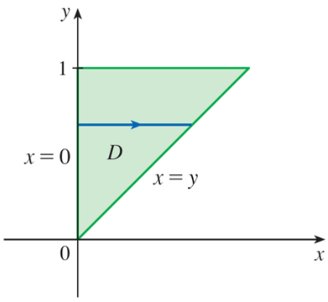
Theorem Let $D = \{(x,y) \mid c \le x \le d,\ h_1(y) \le x \le h_2(y)\}$, where h_1 and h_2 are continuous functions on $[c,d]$. If f is a continuous function on D, then f is integrable on D with the integral of f on D given by

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$$

Examples







(1) Evaluate $\iint_D dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

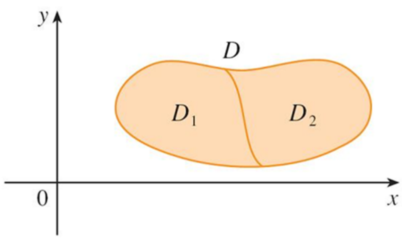
(2) Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

(3) Evaluate $\int_0^1 \int_x^1 e^{y^2}\, dy\, dx$. [Hint: (2) and (3) have the same integration region $D = \{x \le y \le 1,\, 0 \le x \le 1\} = \{0 \le x \le y,\, 0 \le y \le 1\}$]

(4) Evaluate $\int_0^{\pi/2} \int_x^{\pi/2} \frac{\sin y}{y}\, dy\, dx$. [Hint: the integration region $D = \{x \le y \le \pi/2,\, 0 \le x \le \pi/2\} = \{0 \le x \le y,\, 0 \le y \le \pi/2\}$]

Properties of Double Integrals Let $D \subset \mathbb{R}^2$ be a bounded subset, let $f, g : D \to \mathbb{R}$ be integrable functions on D. Then

- $\iint_D [f(x,y) + g(x,y)]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA.$
- $\iint_D cf(x,y)\, dA = c \iint_D f(x,y)\, dA$, where c is a constant.
- if $f(x,y) \ge g(x,y)$ for all $(x,y) \in D$, then
$$\iint_D f(x,y)\, dA \ge \iint_D g(x,y)\, dA.$$
- if $D = D_1 \cup D_1$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then
$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA.$$



- if $f(x,y) = 1$ is integrable on D, then $\iint_D dA = A(D)$, the area of D.

- if $f(x,y)$ is integrable on D and if $m \le f(x,y) \le M$ for all $(x,y) \in D$, then

$$mA(D) \le \iint_D f(x,y)\,dA \le MA(D).$$

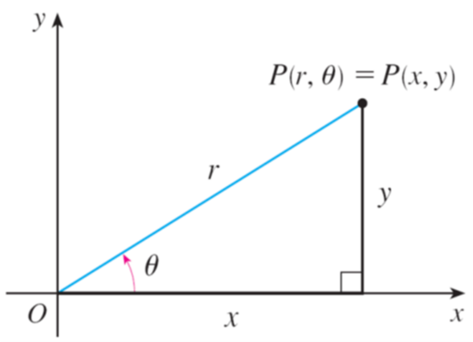
Double Integrals in Polar Coordinates

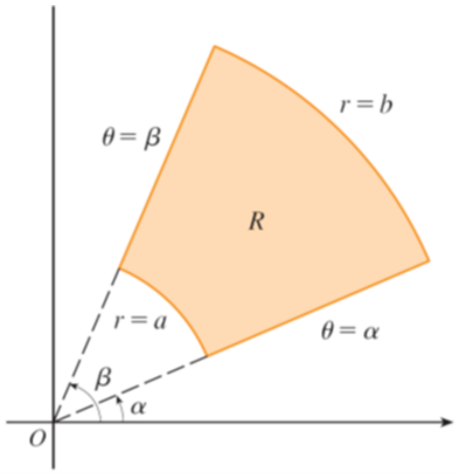
Let P be a point in the xy-plane and let (x,y) and (r,θ) denote the rectangular and polar coordinates of p, respectively. Then (x,y) and (r,θ) are related by the equations

$$r^2 = x^2 + y^2, \quad x = r\cos\theta, \quad y = r\sin\theta.$$

To compute the double integral $\iint_R f(x,y)\,dA$, where R is a polar region of the form

$$R = \{(r,\theta) \mid 0 \le a \le r \le b,\ \alpha \le \theta \le \beta \le 2\pi\},$$





- we divide the interval $[a,b]$ into m subintervals $[r_{i-1}, r_i]$ of equal width $\Delta r_i = \Delta r = \dfrac{b-a}{m}$,
- we divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{j-1}, \theta_j]$ of equal width $\Delta\theta_j = \Delta\theta = \dfrac{\beta-\alpha}{n}$,
- we divide R into $m \times n$ subregions

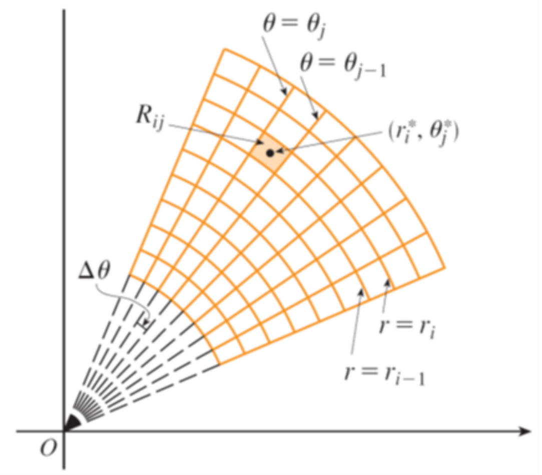
$$R_{ij} = \{(r,\theta) \mid r_{i-1} \le r \le r_i,\ \theta_{j-1} \le \theta \le \theta_j\}$$

with the "center" of the coordinates

$$r_i^* = \frac{r_{i-1}+r_i}{2}, \quad \theta_j^* = \frac{\theta_{j-1}+\theta_j}{2} \quad \Longleftrightarrow \quad (x_i^*, y_j^*) = (r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)$$

and of the area

$$\Delta A_{ij} = A(R_{ij}) = \frac{1}{2}r_i^2(\theta_j - \theta_{j-1}) - \frac{1}{2}r_{i-1}^2(\theta_j - \theta_{j-1}) = \frac{(r_i + r_{i-1})(r_i - r_{i-1})}{2}\Delta\theta = r_i^*\Delta r\Delta\theta$$



Therefore we have

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\,\Delta A_{ij} \\
&= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta \\
&= \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta
\end{aligned}$$

Theorem If f is continuous on $R = \{(r,\theta) \mid 0 \le a \le r \le b,\ \alpha \le \theta \le \beta\}$, where $0 \le \beta - \alpha \le 2\pi$, then

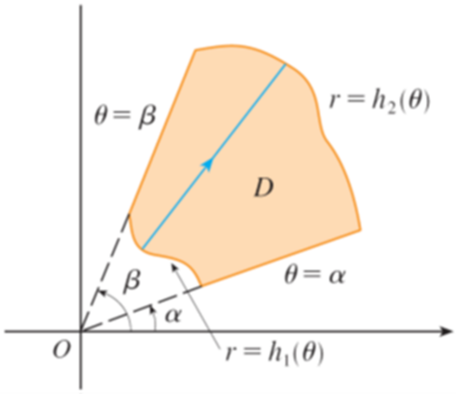
$$\iint_R f(x,y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

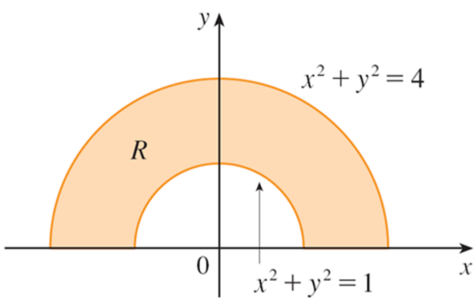
Theorem Let $h_2(\theta) \ge h_1(\theta) \ge 0$ be continuous for each $\theta \in [\alpha, \beta]$. If f is continuous on a polar region of the form

$$D = \{(r,\theta) \mid 0 \le \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\},$$

then

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$





Examples

(1) Evaluate $\iint_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.

(2) Evaluate $\iint_R e^{-x^2-y^2}\,dA$, where $R=(-\infty,\infty)\times(-\infty,\infty)=\{(x,y)\mid -\infty<x,\,y<\infty\}$.

(3) Evaluate $\int_{-\infty}^{\infty} e^{-x^2}\,dx$. Hint: Since $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} e^{-y^2}\,dy$, and x, y are independent variables, we have

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$$

Definition Let $D\subset\mathbb{R}^2$ be a bounded subset with area $A(D)=\iint_D dA$, and let $f:D\to\mathbb{R}$ be an integrable function on D. Then the average value of f over D is defined to be

$$f_{ave}=\frac{1}{A(D)}\iint_D f(x,y)\,dA.$$

Theorem Let $B_r(p)\subset\mathbb{R}^2$ be a disk of radius $r>0$ and center $p\in\mathbb{R}^2$, and let $f:B_r(p)\to\mathbb{R}$ be a continuous function on $B_r(p)$. Then

$$\lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} f(x,y)\,dA = f(p).$$

Proof Since

- $\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} dA = 1$ for all $\rho>0 \implies \lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} dA = 1$,
- $f(p)=f(p)\cdot 1 = f(p)\cdot\lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} dA = \lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\int_{B_\rho(p)} f(p)\,dx$,
- f is continuous at $p \iff \lim_{(x,y)\to p} f(x,y)=f(p)$, that is, for any $\varepsilon>0$, there is a $0<\delta<r$ such that

$$\text{if } (x,y)\in B_\delta(p) \text{ then } |f(x,y)-f(p)|<\varepsilon,$$

so for all $0<\rho<\delta$, we have

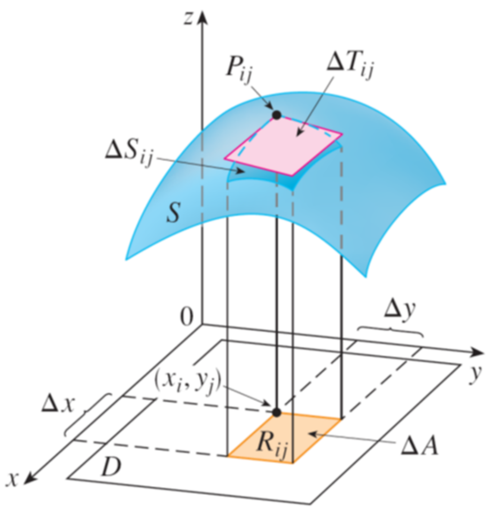
$$\begin{aligned}\left|\lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} f(x,y)\,dA - f(p)\right| &= \left|\lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} [f(x,y)-f(p)]\,dA\right| \\ &\le \lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} |f(x,y)-f(p)|\,dA \\ &< \lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} \varepsilon\,dA \quad \text{since } (x,y)\in B_\rho(p)\subset B_\delta(p) \\ &= \varepsilon.\end{aligned}$$

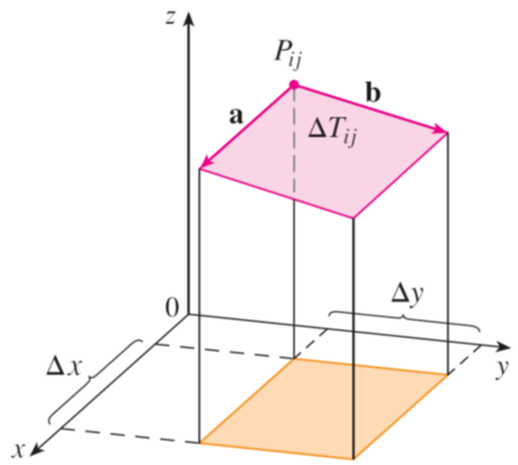
Since $\varepsilon>0$ is arbitrary, we have

$$\lim_{\rho\to 0}\frac{1}{A(B_\rho(p))}\iint_{B_\rho(p)} f(x,y)\,dA = f(p).$$

Surface Area

Let $S = \{(x, y, z) \mid z = f(x, y) \text{ and } (x, y) \in D\}$ be a surface with equation $z = f(x, y)$, where f has continuous partial derivatives. For simplicity in deriving the surface area formula, we assume that $f(x, y) \geq 0$ and the domain D of f is a rectangle.





- Divide the rectangle D into small rectangles R_{ij} with area $\Delta A = \Delta x \Delta y$.
- If (x_i, y_j) is the corner of R_{ij} closest to the origin, let $P_{ij}(x_i, y_j, f(x_i, y_j))$ be the point on S directly above it.
- Approximate the area ΔS_{ij} of the part of S that lies directly above R_{ij} by the area ΔT_{ij} of the part of the tangent plane (a parallelogram) to S at P_{ij} lies directly above R_{ij}.
- Define the surface area of S to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij} \quad \text{if the limit exists,}$$

- Apply the linear (or tangent) approximations at (x_i, y_j) to get

$$(x_i + \Delta x, y_j, f(x_i + \Delta x, y_j)) - (x_i, y_j, f(x_i, y_j)) \approx (\Delta x, 0, f_x(x_i, y_j)\Delta x) = \mathbf{a},$$
$$(x_i, y_j + \Delta y, f(x_i, y_j + \Delta y)) - (x_i, y_j, f(x_i, y_j)) \approx (0, \Delta y, f_y(x_i, y_j)\Delta y) = \mathbf{b},$$

$$\begin{aligned} \Delta T_{ij} &= |\mathbf{a} \times \mathbf{b}| = |\,(-f_x(x_i, y_j),\, -f_y(x_i, y_j),\, 1)\ \Delta x \Delta y| \\ &= \sqrt{[f_x(x_i, y_j)]^2 + [f_y(x_i, y_j)]^2 + 1}\, \Delta x\, \Delta y, \end{aligned}$$

So, if f_x and f_y are continuous on D, then the area of the surface with equation $z = f(x, y)$, $(x, y) \in D$, is given by

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA$$

Example Find the surface area of the part of the surface $z = x^2 + 2y + 2$ that lies above the triangular region $T = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq x\}$ in the xy-plane with vertices $(0, 0)$, $(1, 0)$, and $(1, 1)$.

Triple Integrals

Definition Let $B = [a, b] \times [c, d] \times [r, s] = \{(x, y, z) \in \mathbb{R}^3 \mid a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$ be a closed rectangular box and let f be a function defined on B. Then the triple integral of f over B, denoted $\iiint_B f\, dV$ or simply $\int_B f\, dV$, is defined by

$$\iiint_B f(x, y, z)\, dV = \lim_{\ell, m, n \to \infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V \quad \text{if the limit exists,}$$

$\iff$ For any $\varepsilon > 0$ there is an integer N such that if $\ell,\ m,\ n \ge N,$ then

$$\left| \iiint_B f(x, y)\, dV - \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V \right| < \varepsilon$$

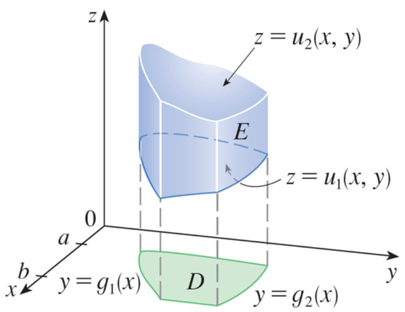
where

- the interval $[a, b]$ is divided into ℓ subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b - a)/\ell$, the interval $[c, d]$ is divided into m subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = (d - c)/m$, and the interval $[r, s]$ is divided into n subintervals $[z_{k-1}, z_k]$ of equal width $\Delta z = (s - r)/n$,
- the rectangular box B is divided into $\ell \times m \times n$ sub-boxes $B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ of equal volume $\Delta V = \Delta x \Delta y \Delta z$,
- $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})$ is an arbitrary point in B_{ijk} and $\sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \Delta V$ is called a triple Riemann sum of f on B.

Fubini's Theorem If f is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then

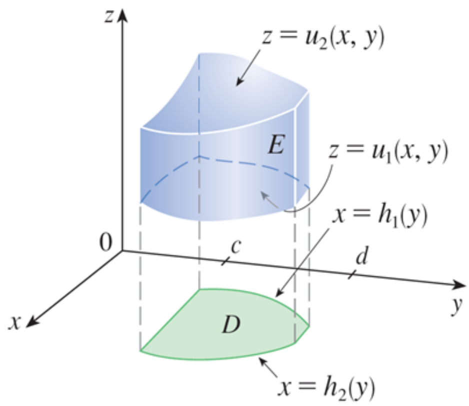
$$\begin{aligned}
\iiint_B f(x, y, z)\, dV &= \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx \\
&= \int_c^d \int_r^s \int_a^b f(x, y, z)\, dx\, dz\, dy = \int_c^d \int_a^b \int_r^s f(x, y, z)\, dz\, dx\, dy \\
&= \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz = \int_r^s \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz
\end{aligned}$$

Example (a) If $E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \le z \le u_2(x, y)\}$, where u_1 and u_2 are continuous functions on D, and $D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, where g_1 and g_2 are continuous functions on $[a, b]$, and if f is continuous on E, then



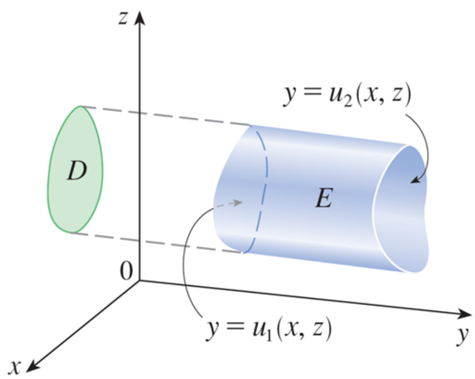
$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx.$$

Example (b) If $E = \{(x,y,z) \mid (x,y) \in D,\ u_1(x,y) \le z \le u_2(x,y)\}$, where u_1 and u_2 are continuous functions on D, and $D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$, where h_1 and h_2 are continuous functions on $[c,d]$, and if f is continuous on E, then



$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.$$

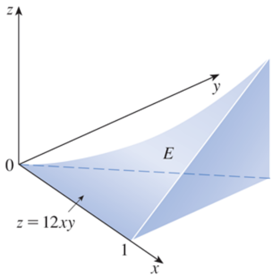
Example (c) If $E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$, where u_1 and u_2 are continuous functions on D, then

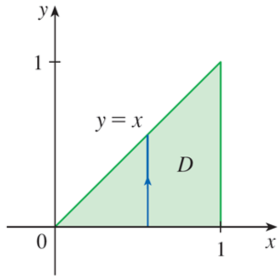


$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA.$$

Examples

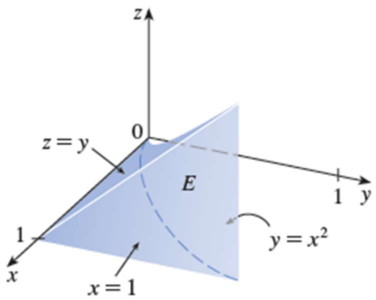
1. Evaluate $\iiint_B xyz^2\,dV$, where $B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$.

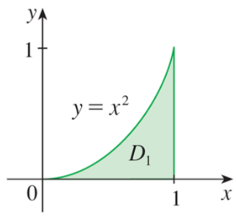


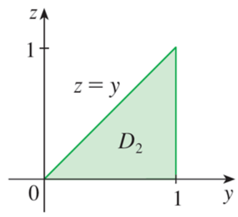


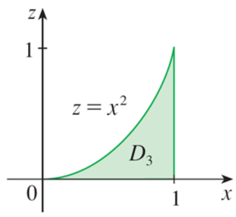
2. Evaluate $\iiint_E z\,dV$, where E is the solid in the first octant bounded by the surface $z = 12xy$ and the planes $y = x$, $x = 1$. [Hint: $E = \{0 \le x \le 1,\ 0 \le y \le x,\ 0 \le z \le 12xy\}$.]

3. Express the iterated integral $\int_0^1 \int_0^{x^2} \int_0^y f(x, y, z)\,dz\,dy\,dx$ as a triple integral and rewrite it in the following order.





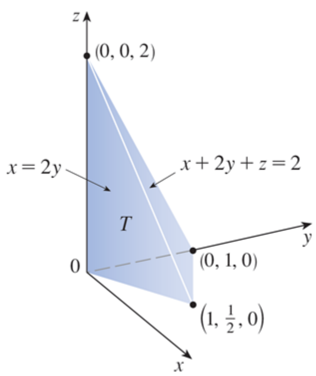


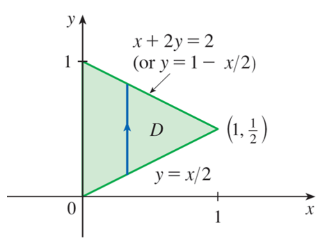


(a) Integrate first with respect to x, then z, and then y. [Solution: $\int_0^1 \int_0^y \int_{\sqrt{z}}^1 f(x, y, z)\,dx\,dz\,dy$]

(b) Integrate first with respect to y, then x, and then z. [Solution: $\int_0^1 \int_{\sqrt{z}}^1 \int_0^{x^2} f(x, y, z)\,dy\,dx\,dz$]

4. Use a triple integral to find the volume of the tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$. [Solution: $V(T) = \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx$]



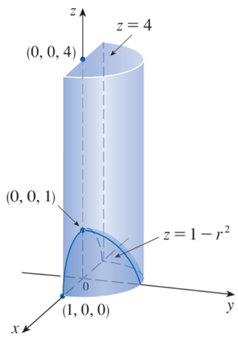


5. Let E be a solid lies within the cylinder $x^2 + y^2 = 1$, to the right of the xz-plane, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$, and let $k > 0$ be a constant and let $\rho(x, y, z) = k\sqrt{x^2 + y^2}$ be the density at $(x, y, z) \in E$.
In terms of the cylindrical coordinates (r, θ, z), E is given by

$$E = \{(r, \theta, z) \mid 0 \le \theta \le \pi,\ 0 \le r \le 1,\ 1 - r^2 \le z \le 4\},$$

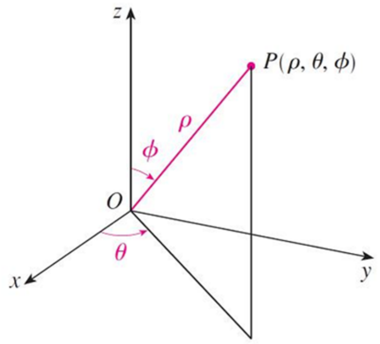
and the (total) mass $m(E)$ of E is given by

$$m(E) = \iiint_E \rho(x, y, z)\,dV = \iiint_E kr\,dV = \int_0^\pi \int_0^1 \int_{1-r^2}^4 kr^2\,dz\,dr\,d\theta.$$



Spherical Coordinates

The spherical coordinates (ρ, θ, ϕ) of a point $P = (x, y, z) \in \mathbb{R}^3$ are defined by the equations



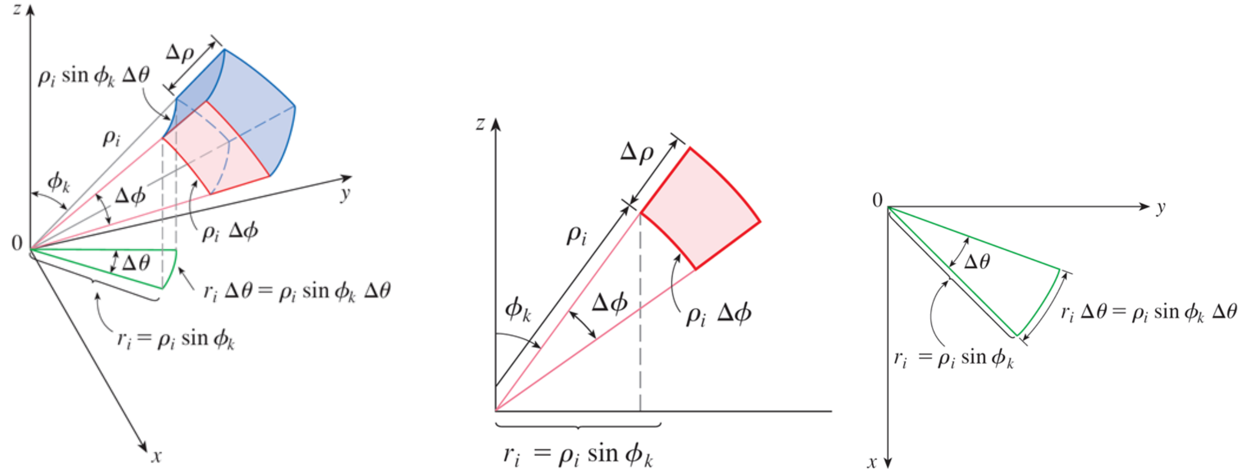
$$x = \rho \sin\phi \cos\theta, \;\; y = \rho \sin\phi \sin\theta, \;\; z = \rho\cos\phi \implies \rho^2 = x^2 + y^2 + z^2,$$

where $\rho \geq 0$ is the distance from P to the origin $O = (0, 0, 0)$, $0 \leq \phi \leq \pi$ is the angle from positive z-axis to $\overline{OP}$, and $0 \leq \theta \leq 2\pi$ is the angle from positive x-axis to the projection of $\overline{OP}$ onto the xy-plane.

Triple Integrals in Spherical Coordinates

To define $\iiint_E f(x, y, z)\, dV$ in the spherical coordinates, we divide E into smaller spherical wedges E_{ijk} by means of equally spaced spheres $\rho = \rho_i$, half-planes $\theta = \theta_j$ and half-cones $\phi = \phi_k$, so that E_{ijk} is approximately a rectangular box with dimensions

- $\Delta\rho = \rho_{i+1} - \rho_i$,
- $\rho_i \Delta\phi =$ arc of a circle with radius ρ_i, angle $\Delta\phi$,
- $\rho_i \sin\phi_k \Delta\theta =$ arc of a circle with radius $\rho_i \sin\phi_k$ and angle $\Delta\theta$.



Then the volume ΔV_{ijk} of E_{ijk} is approximately

$$\Delta V_{ijk} \approx (\Delta\rho)\,(\rho_i\Delta\phi)\,(\rho_i \sin\phi_k\Delta\theta) = \rho_i^2\,\sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi$$

and by choosing a point $(x^*_{ijk}, y^*_{ijk}, z^*_{ijk}) \in E_{ijk}$,

$$\begin{aligned} &\iiint_E f(x,y,z)\,dV \\ =\ & \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x^*_{ijk}, y^*_{ijk}, z^*_{ijk})\Delta V_{ijk} \\ =\ & \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\rho_i^*\sin\tilde\phi_k\cos\tilde\theta_j, \rho_i^*\sin\tilde\phi_k\sin\tilde\theta_j, \rho_i^*\cos\tilde\phi_k)\rho_i^2\,\sin\phi_k\,\Delta\rho\,\Delta\theta\,\Delta\phi \end{aligned}$$

Theorem If E is a spherical wedge

$$E = \{(\rho,\theta,\phi) \mid a \le \rho \le b,\ \alpha \le \theta \le \beta,\ c \le \phi \le d\},$$

where $a \ge 0$, $\beta - \alpha \le 2\pi$ and $d - c \le \pi$, and if f is a continuous function on E, then

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\,\sin\phi\,d\rho\,d\theta\,d\phi.$$

Example Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$. [Hint: $E = \{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4,\ 0 \le \rho \le \cos\phi\}$ and $V(E) = \pi/8$.]

Change of Variables in Multiple Integrals

Recall that

- if f is a continuous function defined on $[a,b]$, $g : [c,d] \to [a,b]$ is a continuously differentiable function with $a = g(c)$, $b = g(d)$, and $g'(u) > 0$ for all $u \in (c,d)$, then

$$\int_a^b f(x)\,dx = \int_c^d f(g(u))\,g'(u)\,du \iff \int_a^b f(x)\,dx = \int_c^d f(x(u))\,\frac{dx}{du}\,du.$$

- S is a bounded closed region in the $r\theta$-plane that corresponds to the region R in the xy-plane by the transformation

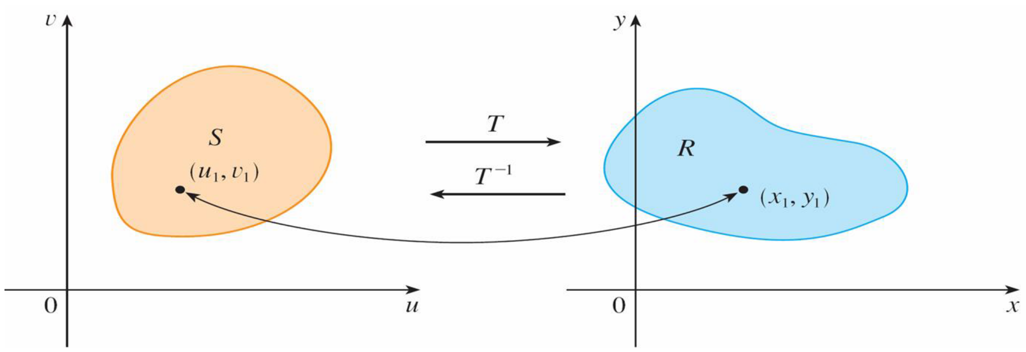
$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \quad \text{for all } (r,\theta) \in S,$$

and if f is a continuous function defined on R, then

$$\iint_R f(x,y)\, dx\, dy = \iint_S f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Change of Variables in Multiple Integrals If S is a bounded closed region in the uv-plane that corresponds to the region R in the xy-plane by a continuously differentiable, one-to-one onto transformation $T : S \to R$ defined by

$$(x,y) = T(u,v) = (x(u,v), y(u,v)) \quad \text{for all } (u,v) \in S.$$



Suppose that f is a continuous function defined on R, then

$$\iint_R f(x,y)\, dx\, dy = \iint_{T(S)} f(x,y)\, dx\, dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv,$$

where $J_T(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)}$ is called the Jacobian of T at (u,v) which is defined by

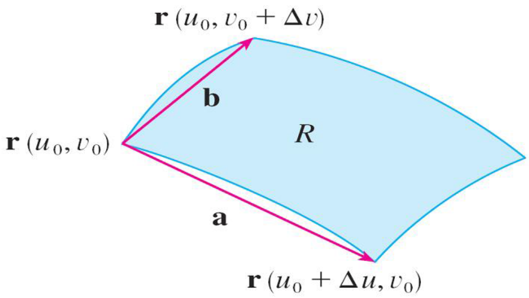
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \text{the determinant of } \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}.$$

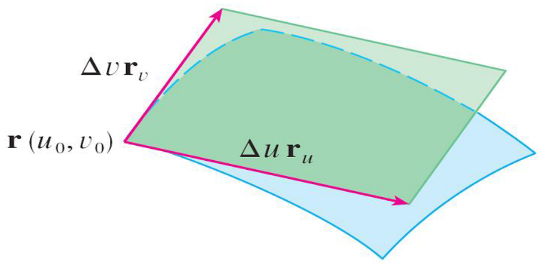
Area of a Parametrized Surface When $R = r(S) \subset \mathbb{R}^3$ is a surface parametrized by the vector function $r : S \to R$, we can approximate the image region $R = r(S)$ by a parallelogram determined by the secant vectors

$$a = r(u_0 + \Delta u, v_0) - r(u_0, v_0), \quad b = r(u_0, v_0 + \Delta v) - r(u_0, v_0).$$

Since

$$r_u = \lim_{\Delta u \to 0} \frac{r(u_0 + \Delta u, v_0) - r(u_0, v_0)}{\Delta u} \implies r(u_0 + \Delta u, v_0) - r(u_0, v_0) \approx \Delta u\, r_u,$$
$$r_v = \lim_{\Delta u \to 0} \frac{r(u_0, v_0 + \Delta v) - r(u_0, v_0)}{\Delta u} \implies r(u_0, v_0 + \Delta v) - r(u_0, v_0) \approx \Delta v\, r_v,$$





we can approximate R by a parallelogram determined by the vectors $\Delta u\, r_u$ and $\Delta v\, r_v$ shown in the following

Therefore we can approximate the area of R by the area of this parallelogram

$$|(\Delta u\, r_u) \times (\Delta v\, r_v)| = |r_u \times r_v|\, \Delta u\, \Delta v$$

where the cross product $r_u \times r_v$ is given by

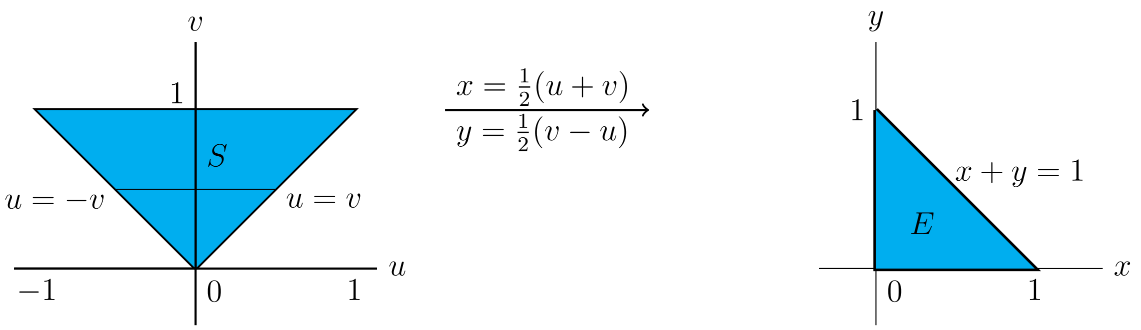
$$r_u \times r_v = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & 0 \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathrm{k} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathrm{k}$$

and the determinant that arises in this calculation is called the Jacobian of the transformation.

Examples

(1) Evaluate

$$\iint_E e^{\frac{x-y}{x+y}}\, dA, \quad \text{where } E = \{(x,y) \mid x \geq 0,\ y \geq 0,\ x+y \leq 1\}.$$



Solution Let $u = x - y$, $v = x + y$. Then $x = \dfrac{u+v}{2}$, $y = \dfrac{v-u}{2}$ and since $(x,y) \in R$ when

$$x \geq 0,\ y \geq 0 \text{ and } x + y \leq 1 \iff u + v \geq 0,\ v - u \geq 0 \text{ and } 0 \leq v \leq 1.$$

If we set

$$S = \{(u,v) \mid u+v \geq 0,\ v-u \geq 0,\ 0 \leq v \leq 1\} = \{(u,v) \mid u \geq -v,\ v \geq u,\ 0 \leq v \leq 1\},$$

we can define $T : S \to R$ by

$$(x,y) = T(u,v) = \Big(\frac{u+v}{2}, \frac{v-u}{2}\Big) \quad \text{for each } (u,v) \in S.$$

Since

$$J_T(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ -\dfrac{1}{2} & \dfrac{1}{2} \end{vmatrix} = \frac{1}{2} \Longrightarrow |J_T(u,v)| = \frac{1}{2},$$

we have

$$\begin{aligned} \iint_R e^{\frac{x-y}{x+y}}\, dA &= \iint_S e^{\frac{x(u,v)-y(u,v)}{x(u,v)+y(u,v)}} \, |J_T(u,v)|\, dA = \int_0^1 \int_{-v}^{v} e^{u/v} \frac{1}{2}\, du\, dv \\ &= \int_0^1 \left(\frac{v}{2} e^{u/v}\right) \Big|_{u=-v}^{u=v} dv = \int_0^1 \frac{v}{2}\left(e - \frac{1}{e}\right) dv \\ &= \frac{v^2}{4}\left(e - \frac{1}{e}\right) \Big|_0^1 = \frac{1}{4}\left(e - \frac{1}{e}\right). \end{aligned}$$

(2) Find the volume V inside the paraboloid $z = x^2 + y^2$ for $0 \le z \le 1$.
Solution Using vertical slices, we see that

$$V = \iint_R (1-z)\, dA = \iint_R \big(1 - (x^2+y^2)\big)\, dA, \text{ where } R = \{(x,y) \mid 0 \le x^2 + y^2 \le 1\}.$$

If we let $S = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ and let $T : S \to R$ be defined by

$$(x,y) = T(r,\theta) = \big(r\, \cos\theta\,,\ r\, \sin\theta\big) \text{ for each } (r,\theta) \in S,$$

then

$$J_T(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \Longrightarrow |J_T(r,\theta)| = r,$$

and

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (1-r^2)\, |J_T(r,\theta)|\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (r - r^3)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^4}{4}\right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{4}\, d\theta = \frac{\pi}{2}. \end{aligned}$$

(3) Find the volume V inside the cone $z = \sqrt{x^2+y^2}$ for $0 \le z \le 1$.
Solution Using vertical slices, we see that

$$V = \iint_R (1-z)\, dA = \iint_R \big(1 - \sqrt{x^2+y^2}\big)\, dA, \text{ where } R = \{(x,y) \mid 0 \le x^2 + y^2 \le 1\}.$$

If we let $S = \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ and let $T : S \to R$ be defined by

$$(x,y) = T(r,\theta) = \big(r\, \cos\theta\,,\ r\, \sin\theta\big) \quad \text{for each } (r,\theta) \in S,$$

then

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^1 (1-r)\, |J_T(r,\theta)|\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (r - r^2)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{r^2}{2} - \frac{r^3}{3}\right) \Big|_0^1 d\theta \\ &= \int_0^{2\pi} \frac{1}{6}\, d\theta = \frac{\pi}{3}. \end{aligned}$$

(4) For $a > 0$, find the volume V inside the sphere $x^2 + y^2 + z^2 = a^2$.
Solution Let

$$\begin{aligned} S &= \{(\rho, \phi, \theta) \mid 0 \le \rho \le a,\ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}, \\ R &= \{(x, y, z) \mid 0 \le x^2 + y^2 + z^2 \le a^2\} \end{aligned}$$

and for each $(\rho, \phi, \theta) \in S$ let $T : S \to R$ be defined by

$$(x, y, z) = T(\rho, \phi, \theta) = \big(\rho\, \sin\phi\, \cos\theta\,,\ \rho\, \sin\phi\, \sin\theta\,,\ \rho\, \cos\phi\big).$$

Then

$$\begin{aligned} J_T(\rho, \phi, \theta) &= \begin{vmatrix} \dfrac{\partial x}{\partial \rho} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial \rho} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial z}{\partial \rho} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\phi\, \cos\theta & \rho\, \cos\phi\, \cos\theta & -\rho\, \sin\phi\, \sin\theta \\ \sin\phi\, \sin\theta & \rho\, \cos\phi\, \sin\theta & \rho\, \sin\phi\, \cos\theta \\ \cos\phi & -\rho\, \sin\phi & 0 \end{vmatrix} \\ &= \cos\phi\big[\rho^2 \sin\phi \cos\phi(\cos^2\theta + \sin^2\theta)\big] + \rho \sin\phi\big[\rho \sin^2\phi(\cos^2\theta + \sin^2\theta)\big] \\ &= \rho^2 \sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2 \sin\phi, \end{aligned}$$

and

$$\begin{aligned} V &= \iiint_S dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} |J_T(\rho, \phi, \theta)|\, d\rho\, d\phi\, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{a} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \left(\frac{\rho^3}{3}\right)\Big|_0^a \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} \frac{a^3}{3}\,(-\cos\phi)\,\Big|_0^{\pi}\, d\theta = \int_0^{2\pi} \frac{2a^3}{3}\, d\theta \\ &= \frac{2a^3\theta}{3}\Big|_0^{2\pi} = \frac{4\pi a^3}{3}. \end{aligned}$$